MATH2050C Selected Solution to Assignment 1

Section 2.1. no 3, 9, 10, 16, 17, 23. Section 2.4. no 1, 2, 5, 8, 9, 10, 13, 14.

Section 2.1.

(2) Prove (a) $-(a+b) = -a-b$. **Solution.** $(a+b)-a-b = (a+b-a)-b = (a-a+b)-b = (0+b)-b = b-b = 0$ by A2, A1, A3, A4. Hence $-a-b$ is the additive inverse to $(a+b)$, that is, $-(a+b) = -a-b$. Note: There could be other proofs.

(3) (b) Solve $x^2 = x$. **Solution.** $x^2 = x$, implies $x^2 - x = x - x = 0$, implies $x(x-1) = 0$, implies $x = 0$, or $x - 1 = 0$, that is, $x = 0$ or $x = 1$.

(5) Prove $1/(ab) = 1/a1/b$ for $a, b \neq 0$. **Solution.** $(ab)1/a1/b = (ab)1/b1/a) = (ab1/b)1/a = (a1)1/a = a1/a = 1$ by M1, M2, M3.

(9) Show that $\mathbb{Q}(\sqrt{2})$ is a field. **Solution.** The multiplicative inverse of $x = p/q + \sqrt{2}m/n$ is given by $(p/q - \sqrt{2}m/n)/((p/q)^2 - 2(m/n)^2)$. Using the fact that $\sqrt{2}$ is irrational, the denominator never vanishes.

(10) (b) Prove that if $0 < a < b$, and $0 \leq c \leq d$, then $0 \leq ac \leq bd$. **Solution.** Sine $c \geq 0$, $0 \leq ac \leq bc$. As $0 \leq c \leq d$ and $b > 0$, $bc \leq bd$, so $0 \leq ac \leq bc \leq bd$, that is, $0 \leq ac \leq bd$.

(16) (c) Solve $1/x < x$. **Solution.** Implicitly it presumes $x \neq 0$. First, if $x > 0$, $1/x < x$ implies $1/xx < xx$, that is, $1 < x^2$ and $x^2 - 1 > 0$. It means $(x+1)(x-1) > 0$. As $x > 0$, we conclude $x > 1$. Next, if $x < 0$, $1/x < x$ implies $1/xx > xx$, that is, $1 > x^2$ or $x^2 - 1 < 0$. From $(x+1)(x-1) < 0$ and $x < 0$ we conclude $-1 < x < 0$. Conclusion: The solution of $1/x < x$ is given by $x > 1$ or $-1 < x < 0$.

(13) Show that $a^2 + b^2 = 0$ if and only of $a = b = 0$ in S. Proof: For a, b in an ordered field S, $x^2 > 0$ for any $x \neq 0$ by Proposition 1.4(a). Also $x^2 \geq 0$ for all x. Hence if $a \neq 0$, then $a^2 + b^2 \geq a^2 > 0$. Similarly, if $b \neq 0$, $a^2 + b^2 \geq b^2 > 0$. Hence $a^2 + b^2 = 0$ forces $a = b = 0$.

(23) For $a, b > 0$, show that $a < b$ iff $a^n < b^n$ for some $n \in \mathbb{N}$. **Solution.** Use induction on n. The conclusion is trivial when $n = 1$. Assume the proposition is true for $n = k$. We have $a^{k+1} = aa^k < ab^k$ (induction hypothesis). As $a < b$, $ab^k < bb^k = b^{k+1}$, so $a^{k+1} < ab^k < b^{k+1}$, the proposition holds for $n = k+1$. The desired conclusion follows.

Section 2.4.

(2) Let $S = \{x = 1/n - 1/m : n, m \in \mathbb{N}\}$. Find $\inf S$ and $\sup S$. **Solution.** For all n, m, $1/n - 1/m > -1/m \geq -1$, so $-1 \leq x$ for all $x \in S$. Moreover, if $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that $n > 1/\varepsilon$ and hence $1/n - 1/1 < -1 + \varepsilon$. Therefore $\inf S = -1$. Similarly, we can show that $\sup S = 1$.

(8) Let f, g be bounded real-valued functions on X. Show that $\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$. **Solution.** For each $x \in X$, $f(x) \leq \sup\{f(z) : z \in X\}$

and $g(x) \le \sup\{g(z) : z \in X\}$ and so $f(x)+g(x) \le \sup\{f(z) : z \in X\} + \sup\{g(z) : z \in X\}$. Taking sup over x, we conclude $\sup\{f(x)+g(x) : x \in X\} \le \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$ after replacing z by x on the right.

(9) **Solution.** We have $\inf_x \sup_y h(x,y) = \sup_y \inf_x = -1$.

(10) **Solution.** We have $\sup_y h(x,y) = 1$, $\inf_x \sup_y h(x,y) = 1$. On the other hand, $\inf_x h(x,y) = 0$, $\sup_y \inf_x h(x,y) = 0$. They are not equal. This exercise tells you that some limits may not be exchangeable.

(14) For $y > 0$, there is some $n \in \mathbb{N}$ such that $0 < 1/2^n < y$. **Solution.** Use induction we prove $n \le 2^n$. By Archimedean Principle, for $x > 0$, there is some n such that $x < n$. Taking $x = 1/y > 0$, $1/y < n < 2^n$, that is, $0 < 1/2^n < y$.

Supplementary Problems

You may use (A1)-(A4), (M1)-(M4), (D), and (O1)-(O3) in the following problems.

(1) Prove for $m,n,p,q \in \mathbb{N}$, $p,q \ne 0$,

$$\frac{n}{m} + \frac{p}{q} = \frac{nq+mp}{mq} .$$

Solution. By (M4) and (D),

$$\frac{n}{m} + \frac{p}{q} = \frac{nq}{mq} + \frac{pm}{qm} = \frac{nq+mp}{mq} .$$

(2) Prove $ab > 0$ implies $a, b > 0$ or $a, b < 0$.

Solution. In case $a = 0$ or $b = 0$, $ab = 0$, so we must have $a, b \ne 0$. If $a > 0, b < 0$, then $a > 0, -b > 0$ by (O3) and $-ab = a(-b) > 0$ by (O2). By (O1) again, $ab < 0$. Hence $a > 0, b < 0$ is excluded. By symmetry, $a < 0, b > 0$ is also excluded.

(3) Prove $2ab \le a^2 + b^2$.

Solution. First we claim $x^2 > 0$ for $x \ne 0$. When $x > 0$, it follows from (O2). When $x < 0$, $(-x)(x-x) = (-x)0 = 0$ implies $-x^2 + (-x)^2 = 0$, that is, $x^2 = (-x)^2 > 0$. Assume $a \ne b$. Letting $x = a - b$, $(a-b)^2 > 0$ which is just $a^2 + b^2 > 2ab$. When $a = b$, $a^2 + b^2 = 2ab$. We conclude equality holds in this inequality iff $a = b$.

(5) Let p be a prime number. Show that $\mathbb{Z}_p$ is a field but it does not admit an ordering satisfying (O1)-(O3).

Solution. We define $a + b$ and ab by modulo p. For $a \in \mathbb{Z}_p = \{0, 1, 2, \cdots, p-1\}$, the additive inverse of a is $p - a$ (so that $a + (p-a) = p \equiv 0$). We claim for $a \in \mathbb{Z}_p$, there is some $b \in \mathbb{Z}_p$ such that $ab \equiv 1$. For, consider the set $\{0, a, 2a, \cdots, (p-1)a\}$. After taking modulo p, this set consists of p many elements and they are all distinct (because p is prime). Therefore, it is equal to $\{0, 1, 2, \cdots, p-1\}$, so there must exist a unique b such that $ba \equiv 1$, that is, b is the multiplicative inverse of a.

However, $\mathbb{Z}_p$ is not an ordered field. For, adding 1 p many times $1 + \cdots + 1 = p \equiv 0$. If it is ordered, we have $p > 0$ by Proposition 1.4(c).

(6) Is $\mathbb{C}$ an ordered field?

Solution. NO. Suppose it admits an ordering $<$. By Proposition 1.4(a), $i^2 > 0$. But then it means $-1 > 0$ or $1 < 0$, contradicting Proposition 1.4(a) $1^2 = 1 > 0$.